Algebras associated to separated graphs

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PARS, Gramado, May 2014

Separated graphs: the initial motivation

Leavitt (1962) defined algebras $L_K(m,n)$ for $1 \leq m \leq n$ in the following way:
$L_K(m,n)$ is the K-algebra with generators

$$\{X_{ji}, X_{ji}^* : 1 \leq j \leq m, 1 \leq i \leq n\}$$

and defining relations:

$$XX^* = I_m, \quad X^*X = I_n,$$

where $X = (X_{ji})$.

Separated graphs

Definition

A *separated graph* is a pair (E, C) where E is a graph, $C = \bigsqcup_{v \in E^0} C_v$, and C_v is a partition of $r^{-1}(v)$ (into pairwise disjoint nonempty subsets) for every vertex v:

$$r^{-1}(v) = \bigsqcup_{X \in C_v} X.$$

(In case v is a source, we take C_v to be the empty family of subsets of $r^{-1}(v)$.)
The constructions we introduce revert to existing ones in case $C_v = \{r^{-1}(v)\}$ for each $v \in E^0$. We refer to a *non-separated graph* in that situation.

The Leavitt path algebra of a separated graph

Definition

The *Leavitt path algebra of the separated graph* (E, C) with coefficients in the field K, is the K-algebra $L_K(E, C)$ with generators $\{v, e, e^* \mid v \in E^0, e \in E^1\}$, subject to the following relations:

(V) $vv' = \delta_{v,v'} v$ for all $v, v' \in E^0$,

(E1) $r(e)e = e = es(e)$ for all $e \in E^1$,

(E2) $s(e)e^* = e^* = e^* r(e)$ for all $e \in E^1$,

(SCK1) $e^* e' = \delta_{e,e'} s(e)$ for all $e, e' \in X$, $X \in C$, and

(SCK2) $v = \sum_{e \in X} ee^*$ for every finite set $X \in C_v$, $v \in E^0$.

Example

Let $1 \leq m \leq n$. Let us consider the separated graph $(E(m,n), C(m,n))$, where $E(m,n)$ is the graph consisting of two vertices v, w and with

$$E(m,n)^1 = \{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\},$$

with $s(\alpha_i) = s(\beta_j) = v$ and $r(\alpha_i) = r(\beta_j) = w$ for all i, j, and $C(m,n)$ consists of two elements $X = \{\alpha_1, \dots, \alpha_n\}$ and $Y = \{\beta_1, \dots, \beta_m\}$.

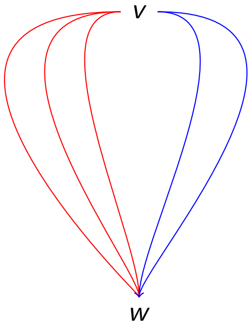

Figure: The separated graph $(E(2,3), C(2,3))$

Lemma (E. Pardo)

There is a natural isomorphism

$$\gamma \colon L_K(m,n) \to wL_K(E(m,n), C(m,n))w$$

given by

$$\gamma(X_{ji}) = \beta_j^* \alpha_i, \quad \gamma(X_{ji}^*) = \alpha_i^* \beta_j.$$

This induces an isomorphism

$$L_K(E(m,n), C(m,n)) \cong M_{n+1}(L_K(m,n)) \cong M_{m+1}(L_K(m,n)).$$

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Note that

$$\gamma(\sum_{i=1}^{n} X_{ji}X_{ki}^*) = \sum_{i=1}^{n} \beta_j^*\alpha_i\alpha_i^*\beta_k = \beta_j^*\beta_k = \delta_{jk}w$$

and similarly $\gamma(\sum_{j=1}^{m} X_{ji}^* X_{jk}) = \delta_{ik}w$ so γ is a well-defined homomorphism, which is shown to be an isomorphism.

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Definition

Let (E, C) be a finitely separated graph. The *monoid* of (E, C) is the abelian monoid $M(E, C)$ with generators $\{a_v \mid v \in E^0\}$ and relations

$$a_v = \sum_{e \in X} a_{r(e)}, \qquad \forall X \in C_v, \forall v \in E^0.$$

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Theorem (Goodearl-A)

If (E, C) is a finitely separated graph then the natural map

$$M(E, C) \to \mathcal{V}(L_K(E, C))$$

is an isomorphism.

Example

For $(E, C) = (E(m, n), C(m, n))$, we have

$$\mathcal{V}(L(E, C)) \cong M(E, C) \cong \langle a \mid ma = na \rangle.$$

a result originally due to Bergman.

Definition

For any separated graph (E, C), the (full) graph C*-algebra of the separated graph (E, C) is the universal C*-algebra with generators $\{v, e \mid v \in E^0, \ e \in E^1\}$, subject to the following relations:

(V) $vw = \delta_{v,w} v$ and $v = v^*$ for all $v, w \in E^0$,

(E) $r(e)e = e = es(e)$ for all $e \in E^1$,

(SCK1) $e^* f = \delta_{e,f} s(e)$ for all $e, f \in X$, $X \in C$, and

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In case (E, C) is trivially separated, $C^*(E, C)$ is just the classical graph C*-algebra $C^*(E)$.

Graph C*-algebras and dynamics

It is well-known that graph C*-algebras (of ordinary graphs) are closely related to dynamics. This was first discovered by Cuntz and Krieger for $\mathcal{O}_n$ and related C*-algebras $\mathcal{O}_A$, nowadays known as Cuntz-Krieger C*-algebras.

In particular $\mathcal{O}_n$ is related to the shift on $X = \{1, \dots, n\}^{\mathbb{N}}$.

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In particular $\mathcal{O}_n$ is related to the shift on $X = \{1, \dots, n\}^{\mathbb{N}}$.

Note that $X = \bigsqcup_{i=1}^n H_i$, with $X \cong H_i$ for all i.
$(H_i = \{(i, x_2, x_3, \dots,)\}.)$
We extend this to the case (m, n), as follows:

Dynamical systems of type (m,n)

We study pairs of compact Hausdorff topological spaces (X, Y) such that

$$X = \bigcup_{i=1}^{n} H_i = \bigcup_{j=1}^{m} V_j,$$

where the H_i are pairwise disjoint clopen subsets of X, each of which is homeomorphic to Y via given homeomorphisms $h_i : Y \to H_i$. Likewise we will assume that the V_i are pairwise disjoint clopen subsets of X, each of which is homeomorphic to Y via given homeomorphisms $v_i : Y \to V_i$.

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Definition

We will refer to the quadruple $\left(X, Y, \{h_i\}_{i=1}^{n}, \{v_j\}_{j=1}^{m}\right)$ as an (m, n)-dynamical system.

Definition

An (m,n)-dynamical system $\left(X^u, Y^u, \{h_i^u\}_{i=1}^n, \{v_j^u\}_{j=1}^m\right)$ is *universal* if it satisfies the following condition: given any (m,n)-dynamical system

$$\left(X, Y, \{h_i\}_{i=1}^n, \{v_j\}_{j=1}^m\right),$$

there exists a unique continuous map

$$\gamma : \Omega = X \bigsqcup Y \to \Omega^u = X^u \bigsqcup Y^u,$$

such that

1. $\gamma(Y) \subseteq Y^u$,
2. $\gamma(X) \subseteq X^u$,
3. $\gamma \circ h_i = h_i^u \circ \gamma$,
4. $\gamma \circ v_j = v_j^u \circ \gamma$.

Example

When $m = 1$, the universal $(1, n)$ dynamical system consists of $X^u = \{1, \dots, n\}^{\mathbb{N}}$, $Y^u = \{1', \dots, n'\}^{\mathbb{N}}$, a disjoint copy of X^u, $X^u = \bigcup_{i=1}^n H_i$, where

$$H_i = \{(i, x_2, x_3, \dots,) : x_n \in \{1, \dots, n\}\},$$

$h_i \colon Y^u \to X^u$ sends $(x_1', x_2', \dots)$ to $(i, x_1, x_2, \dots)$, and $v \colon Y^u \to X^u$ sends $(x_1', x_2', \dots)$ to $(x_1, x_2, \dots)$.

In general, the universal (m, n) dynamical system is related to the graph C*-algebra $A_{m,n} := C^*(E(m,n), C(m,n))$, as follows:

Definition

Let U be the subset of partial isometries in $A_{m,n}$ given by

$$U = \{\alpha_1, \dots, \alpha_n,\ \beta_1, \dots, \beta_m\}.$$

We will let $\mathcal{O}_{m,n}$ be the quotient of $A_{m,n}$ by the closed two-sided ideal generated by all elements of the form

$$xx^*x - x,$$

as x runs in $\langle U \cup U^* \rangle$.

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It is worth to mention that $A_{1,n} = \mathcal{O}_{1,n} \cong M_2(\mathcal{O}_n)$, because $\alpha_1, \dots, \alpha_n, \beta_1$ is a *tame set* of partial isometries when $m = 1$.

Note that there is a *partial action* θ of $\mathbb{F}_{n+m}$, the free group on $\{a_1, \dots, a_n, b_1, \dots, b_m\}$ on $\Omega^u = X^u \bigsqcup Y^u$, obtained by sending a_i to h_i and b_j to v_j.

Theorem

There is a natural isomorphism

$$\mathcal{O}_{m,n} \cong C(\Omega^u) \rtimes_{\theta^*} \mathbb{F}_{n+m},$$

where $C(\Omega^u) \rtimes_{\theta^*} \mathbb{F}_{n+m}$ denotes the crossed product of the C-algebra $C(\Omega^u)$ by the induced partial action θ^* of $\mathbb{F}_{n+m}$.*

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All the above can be generalized to any finite bipartite separated graph (E, C), obtaining C*-algebras $\mathcal{O}(E, C)$ which are suitable full crossed products of commutative C*-algebras by partial actions of free groups.

The algebra $L_K^{\mathrm{ab}}(E, C)$

The theory is very similar in the purely algebraic case. Let (E, C) be as before. We look at the construction in some detail:

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Set $U = \langle E^1 \cup (E^1)^* \rangle$, the multiplicative semigroup of $L_K(E, C)$ generated by $E^1 \cup (E^1)^*$. For $u \in U$ set $e(u) = uu^*$ (not an idempotent in general). Write

$$L_K^{\mathrm{ab}}(E, C) = L_K(E, C) / \langle [e(u), e(u')] : u, u' \in U \rangle.$$

It can be shown that $\{\overline{e(u)} : u \in U\}$ is a family of commuting *idempotents* in $L_K^{\mathrm{ab}}(E, C)$.

Let $\mathcal{B}$ be the commutative subalgebra of $L_K^{\mathrm{ab}}(E, C)$ generated by the idempotents $\overline{e(u)}$, for $u \in U$.

There exists a totally disconnected, metrizable, compact space $\Omega(E, C)$ such that

$$\mathcal{B} = C_K(\Omega(E, C)),$$

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There exists a totally disconnected, metrizable, compact space $\Omega(E,C)$ such that

$$\mathcal{B} = C_K(\Omega(E,C)),$$

where $C_K(\Omega)$ denotes the algebra of locally constant functions $\Omega \to K$. Moreover there is a partial action α of $\mathbb{F} = \mathbb{F}\langle E^1 \rangle$ on $\mathcal{B}$ (given essentially by conjugation) which induces a partial action α^* by homeomorphisms of $\mathbb{F}$ on $\Omega(E,C)$. Moreover, we show:

Theorem

$$L_K^{\mathrm{ab}}(E,C) \cong C_K(\Omega(E,C)) \rtimes_\alpha \mathbb{F}.$$

Example

Our first example is the "pure refinement" example. Let (E, C) be as in the picture, with $C_v = \{X, Y\}$ and $X = \{\alpha_1, \alpha_2\}$, $Y = \{\beta_1, \beta_2\}$. The corner $vL_K(E, C)v$ is isomorphic to a free product $K^2 *_K K^2$. The corner of the abelianized Leavitt algebra $L_K^{\mathrm{ab}}(E, C)$ is isomorphic to K^4. We thus see a drastic reduction of the complexity in the transition from $L(E, C)$ to $L^{\mathrm{ab}}(E, C)$. A similar statement holds for the C*-algebras $C^*(E, C)$ and $\mathcal{O}(E, C)$.

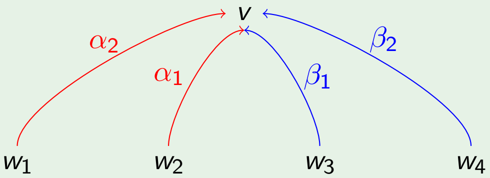

Figure: The separated graph of pure refinement

Example

Let (E, C) be the separated graph described in the Figure, with $C_v = \{X, Y\}$ and $X = \{\alpha_0, \alpha_1\}$ and $Y = \{\beta_0, \beta_1\}$.

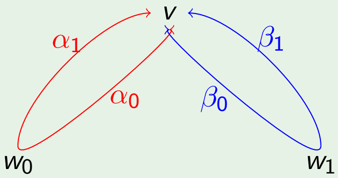

Figure: The separated graph corresponding to Truss example

The Leavitt path algebra $L_K(E, C)$ is Morita equivalent to the corner $vL(E, C)v$, which is isomorphic to $M_2(K) *_K M_2(K)$.

$L_K^{\mathrm{ab}}(E,C) \cong C_K(\Omega(E,C)) \rtimes \mathbb{F}_4$, where $\Omega := \Omega(E,C)$ is a 0-dimensional compact space admitting a decomposition $\Omega = X \sqcup Y \sqcup Z$ into clopen subsets, such that Z decomposes in two different ways as a disjoint union of clopen subsets $Z = H_0 \sqcup H_1 = V_0 \sqcup V_1$, together with homeomorphisms $\mathfrak{h}_i \colon X \to H_i$ for $i = 0, 1$, and $\mathfrak{v}_j \colon Y \to V_j$ for $j = 0, 1$.

This construction is closely related to an example due to Truss, defined using other tools.
Note that

$$2[X] = [Z] = 2[Y]$$

and Truss showed that $[X] \neq [Y]$ in the type semigroup.

Example

Let (E, C) be the separated graph described in the Figure, with $C_v = \{X, Y\}$ and $X = \{\alpha_1, \alpha_2\}$ and $Y = \{\beta_1, \beta_2\}$.
The corner $vC^*(E, C)v$ is the unital universal C*-algebra generated by a partial isometry (studied recently by Brenken and Niu).

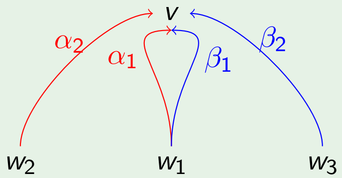

Figure: The separated graph of a partial isometry

The C*-algebra $v\mathcal{O}(E,C)v$ is the C*-algebra of the *free monogenic inverse monoid*. The 0-dimensional compact space $\Omega(E,C)$ decomposes as $\Omega(E,C) = X_v \sqcup \bigsqcup_{i=1}^{3} Y_{w_i}$, and we have two decompositions $X_v = H_{\beta_1} \sqcup H_{\beta_2} = H_{\alpha_1} \sqcup H_{\alpha_2}$ in clopen sets, with a universal homeomorphism $\alpha := \theta_{\alpha_1} \circ \theta_{\beta_1}^{-1} \colon H_{\beta_1} \to H_{\alpha_1}$, in the sense that given any other homeomorphism $\beta \colon X_1' \to Z_1'$, where X_1' and Z_1' are clopen subsets of a compact Hausdorff space X', there exists a unique equivariant continuous map from X' to X_v.

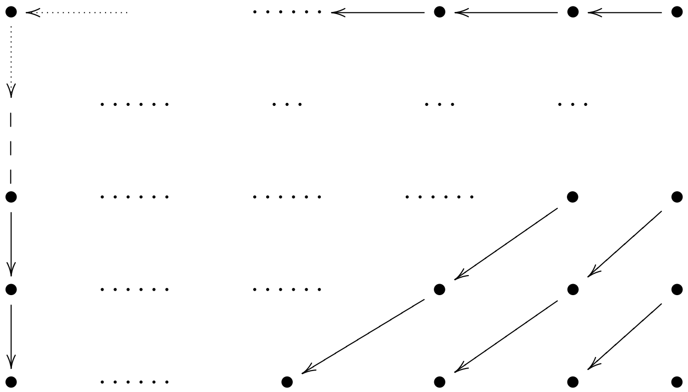

Figure: The compact space X_v.

Example

Let (F, D) be as in the Figure, with $C_v = \{X, Y\}$ and $X = \{\alpha_1, \alpha_2\}$ and $Y = \{\beta_1, \beta_2\}$. We have

$$vC^*(E, C)v \cong C^*((*_{\mathbb{Z}}\mathbb{Z}_2) \rtimes \mathbb{Z}) \quad , vL(E, C)vK[(*_{\mathbb{Z}}\mathbb{Z}_2) \rtimes \mathbb{Z}],$$

where $\mathbb{Z}$ acts on $*_{\mathbb{Z}}\mathbb{Z}_2$ by shifting the factors of the free product.

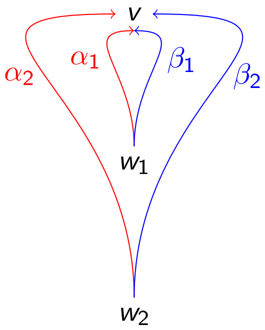

Figure: The separated graph underlying the lamplighter group

It is easy to show that

$$v\mathcal{O}(F,D)v \cong C^*(\mathbb{Z}_2 \wr \mathbb{Z}), \qquad vL_K^{\mathrm{ab}}(F,D)v \cong K[\mathbb{Z}_2 \wr \mathbb{Z}],$$

where $\mathbb{Z}_2 \wr Z$ is the wreath product $(\oplus_{\mathbb{Z}}\mathbb{Z}_2) \rtimes \mathbb{Z}$. This is a well-known group, called the lamplighter group. This group provided the first counter-example to the Strong Atiyah's Conjecture.

Our results produce in this case the well-known representation of the group algebra of the lamplighter group:

$$K[\mathbb{Z}_2 \wr \mathbb{Z}] \cong vL_K^{\mathrm{ab}}(F,D)v \cong C_K\Big(\prod_{\mathbb{Z}} \mathbb{Z}_2\Big) \rtimes \mathbb{Z}.$$

This algebra is the algebra associated to the bilateral shift on $\{0,1\}^{\mathbb{Z}}$.

Applications

- Paradoxical decompositions.
- The realization problem for von Neumann regular rings.

Paradoxical decompositions

Let G be a group acting on a set X.
$E, E' \subseteq X$ are **equidecomposable** if

$$E = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n, \quad E' = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_n$$

and there exist $g_1, g_2, \dots, g_n \in G$ such that $B_i = g_i A_i$ for all $i = 1, \dots, n$.

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The *type semigroup* $S(X, G)$ is defined by using this relation. Elements of $S(X, G)$ are finite sums of equidecomposability classes $[E]$, for $E \subseteq X$.

A subset $E \subseteq X$ is called **paradoxical** if $E_1 \sqcup E_2 \subseteq E$ with $E_1 \sim_G E$ and $E_2 \sim_G E$.

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Note that $E \subseteq X$ is paradoxical $\iff 2[E] \leq [E]$ in $S(X, G)$.

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The Banach-Tarski Theorem (or Paradox) asserts that the unit ball $\mathbb{B}^1$ is $\mathbb{G}$-paradoxical, where $\mathbb{G}$ is the group of all the isometries of $\mathbb{R}^3$.

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The study of this concept led to the notion of **amenable group**: A discrete group Γ is **amenable** if ${}_\Gamma\Gamma$ is not paradoxical.

Tarski's Theorem

Theorem (Tarski)

Let G be a group acting on a set X. Then the following conditions are equivalent:

1. *E is not G-paradoxical, i.e.* $2[E] \not\lesssim [E]$
2. *There exists a finitely additive G-invariant measure* $\mu\colon \mathcal{P}(X) \to [0, +\infty]$ *such that* $\mu(E) = 1$.

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This result gives the transition from the paradoxical decompositions characterization of amenable groups to other characterizations, notably the one involving invariant means.

About the proof

The proof of Tarski's Theorem is based on the purely semigroup theoretic result:

Theorem

Let $(S, +)$ be an abelian semigroup and $e \in S$. Then the following are equivalent:

(a) *There exists a semigroup homomorphism $\mu \colon S \to [0, \infty]$ such that $\mu(e) = 1$.*

(b) *For all $n \in \mathbb{N}$, we have $(n+1)e \not\leq ne$.*

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(b) *For all $n \in \mathbb{N}$, we have $(n+1)e \not\leq ne$.*

and the following properties of $S(X,G)$:

Schröder-Bernstein axiom: $a \leq b$ and $b \leq a \implies a = b$.

Cancellation law: $\forall n \in \mathbb{N}, \quad na = nb \implies a = b$.

In fact, with these conditions at hand we can easily show that condition (b) in the Theorem is equivalent to $2e \nleq e$, or equivalently

$$2e \leq e \iff (n+1)e \leq ne \quad \text{for some n.}$$

If $(n+1)e \leq ne$ then $(n+1)e = ne$ by Schröder-Bernstein, and then

$$(n+1)e = ne \implies n(2e) = ne \implies 2e = e \quad \text{by the cancellation law.}$$

There has been recent interest in trying to extend Tarski's theorem to a more general context:

Assume that G acts on a set X and let $\mathbb{D}$ be a G-invariant subalgebra of sets of X. Then one can restrict the G-equidecomposability relation to elements of $\mathbb{D}$, and obtain a type semigroup $S(X, G, \mathbb{D})$.

In recent papers by Rørdam–Sierakowski and Kerr–Nowak, the following particular case has been considered:

G acts by homeomorphisms on a totally disconnected compact Hausdorff space X (e.g. the Cantor set) and $\mathbb{D}$ is the subalgebra $\mathbb{K}$ of clopen subsets of X.

These authors have raised the question of whether the analogue of Tarski's Theorem holds in this context. More precisely:

Is it true that, for $E \in \mathbb{K}$, one has that the following are equivalent?

(1) $2[E] \not\leq [E]$ in $S(X, G, \mathbb{K})$,
(2) There exists a semigroup homomorphism $\mu\colon S(X, G, \mathbb{K}) \to [0, \infty]$ such that $\mu([E]) = 1$.

One may ask:

What are the general properties of $S(X, G, \mathbb{K})$? It is easy to show that $S(X, G, \mathbb{K})$ has the following properties:

• It is **conical** $x + y = 0 \implies x = y = 0$

• It has the **Riesz refinement property**: If $a + b = c + d$ then $\exists x, y, z, t$ such that $a = x + y$, $b = z + t$, $c = x + z$ and $d = y + t$:

	c	d
a	x	y
b	z	t

We prove that these are the only general properties of $S(X, G, \mathbb{K})$:

Theorem

Let M be an arbitrary f.g. conical abelian monoid. Then there exists a totally disconnected, metrizable compact space X and an action of a finitely generated free group $\mathbb{F}$ on it such that there is an order-embedding $M \hookrightarrow S(X, \mathbb{F}, \mathbb{K})$.

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For instance, taking $M = \langle a \mid na = ma \rangle$ for $1 < m < n$ one obtains that there is a clopen subset $E \subseteq X$ such that $2[E] \nleq [E]$ in $S(X, \mathbb{F}, \mathbb{K})$, but there is no $\mu\colon S(X, \mathbb{F}, \mathbb{K}) \to [0, \infty]$ such that $\mu([E]) = 1$.

In the general setting of a partial action θ of a group Γ on a totally disconnected compact space X, we always have a monoid homomorphism:

$$S(X,\Gamma,\mathbb{K}) \longrightarrow \mathcal{V}(C_K(X) \rtimes_{\theta^*} \Gamma)$$

$$[Y] \mapsto \chi_Y \cdot \delta_e$$

If $X = \Omega(E,C)$ for a finite bipartite separated graph (E,C), we are able to show:

Theorem

The natural homomorphism

$$S(\Omega(E,C),\mathbb{F},\mathbb{K}) \longrightarrow \mathcal{V}(C_K(\Omega(E,C)) \rtimes_{\alpha} \mathbb{F})$$

is an isomorphism

Now, starting with a finitely generated conical abelian monoid M, we choose a finite bipartite separated graph (E, C) such that $M \cong M(E, C)$, and so we get a totally disconnected metrizable compact space $\Omega(E, C)$ with a partial action α^* of $\mathbb{F} = \mathbb{F}\langle E^1 \rangle$ such that there is an order-embedding

$$M \hookrightarrow \mathcal{V}(L^{\mathrm{ab}}(E, C)) \cong S(\Omega(E, C), \mathbb{F}, \mathbb{K}).$$

Finally, using globalization techniques due to Abadie, we can reach the same conclusion, but with *total actions* instead of *partial actions*, obtaining:

Theorem

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Corollary

There exist a global action of a finitely generated free group $\mathbb{F}$ on a totally disconnected metrizable compact space Z, and a non-$\mathbb{F}$-paradoxical (with respect to $\mathbb{K}$) clopen subset A of Z such that $\mu(A) = \infty$ for every finitely additive $\mathbb{F}$-invariant measure $\mu\colon \mathbb{K} \to [0, \infty]$ such that $\mu(A) > 0$.

The realization problem for von Neumann regular rings

Definition

A ring R is said to be *von Neumann regular* if $\forall a \in R \; \exists b \in R$ such that $a = aba$.

The realization problem for von Neumann regular rings asks whether all the countable conical refinement monoids appear as monoids $\mathcal{V}(R)$ for some regular ring R.

In a joint paper with Brustenga, the monoids of the form $M(E)$, for a non-separated graph E, where realized by a certain 'algebra of fractions' $Q_K(E)$ of the Leavitt path algebra $L_K(E)$.

In a paper with Goodearl, we have investigated the problem for the refinement monoid M arising from the group algebra of the free monogenic monoid:

Theorem

If K is an uncountable field, then there is no regular K-algebra R such that $\mathcal{V}(R) \cong M$. If K is a countable field then there is a regular K-algebra R with $\mathcal{V}(R) \cong M$.

In a paper with Goodearl, we have investigated the problem for the refinement monoid M arising from the group algebra of the free monogenic monoid:

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The monoid M is presented by generators

$$x_0, y_0, z_0, a_1, x_1, y_1, z_1, a_2, x_2, y_2, z_2, \dots$$

and relations

$$x_0 + y_0 = x_0 + z_0\,, \qquad y_l = y_{l+1} + a_{l+1}\,, \qquad z_l = z_{l+1} + a_{l+1}\,,$$
$$x_l = x_{l+1} + y_{l+1} = x_{l+1} + z_{l+1}\,.$$

P. Ara, K. R. Goodearl, *C*-algebras of separated graphs*, J. Funct. Anal. **261** (2011), 2540–2568.

P. Ara, K. R. Goodearl, *Leavitt path algebras of separated graphs*, Crelle's journal, **669** (2012), 165–224.

P. Ara, *Purely infinite simple reduced graph C*-algebras of one-relator graphs*, J. Math. Anal. Appl. **393** (2012), 493–508.

P. Ara, R. Exel, T. Katsura, Dynamical systems of type (m, n) and their associated C*-algebras, Ergodic Theory and Dyn. Systems **33** (2013), 1291–1325.

P. Ara, R. Exel, Dynamical systems associated to separated graphs, graph algebras, and paradoxical decompositions, Advances in Mathematics, **252** (2014), 748–804.